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TOWARDS A WAVE THEORY OF CHARGED BEAM TRANSPORT: A COLLECTION OF THOUGHTS

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ABSTRACT

In this paper we formulate in a rigorous way a wave theory of charged beam linear transport. The Wigner distribution function is introduced and provides the link with classical mechanics. Finally the Von-Neumann equation is shown to coincide with the Liouville equation for the linear transport.

INTRODUCTION

A formal "quantum" theory of charged beam transport has been recently proposed. (Ref. 1) Within such a context the possibility of viewing the beam emittance as a kind of quantization constant has been considered on the basis of some strong conceptual similarities existing with the so-called "quantum" theory of light rays. (Ref. 2)

The proposed quantization procedure (described in some detail below) cannot be thought as fully satisfactory and in particular the role played by the "beam wave function" (b.w.f.) and its link with classical dynamics have not ben thoroughly investigated and clarified.

In this paper we develop, in a more rigorous way, the fundamental steps towards a "quantum" theory of beam transport through linear elements, showing the relation with classical mechanics and discussing the connections with the Liouville equation which describes the time evolution of non-interacting classical ensembles. We will show that classical dynamics and the Courant-Snyder theory (Ref. 3) can be recovered by using the formalism of the Wigner phase-space function, (Ref. 4) which therefore can be seen as the natural framework to study the evolution of charged beam transport through linear magnetic systems.

The paper is organized as follows. In Sect. 1 we review some aspects of symplectic mechanics (Ref. 5) and introduce some definitions which will be useful in Sect. 2 where the "quantum" theory of charged linear transport will be completely developed in terms of the Wigner phase-space function. Some comments and final remarks on the "quantum" theory of nonlinear charged transport conclude this paper.

1. SYMPLECTIC MECHANICS OF QUADRATIC HAMILTONIANS

The most general time dependent quadratic Hamiltonian in one degree of freedom can be written as

$$H = \frac{1}{2}a(t)q^2 + \frac{1}{2}b(t)p^2 + c(t)qp \tag{1.1}$$

or in a matrix form as

$$H = \frac{1}{2}\underline{z}^T \hat{H}\,\underline{z} \tag{1.2}$$

where

$$\underline{z} = \begin{pmatrix} q \\ p \end{pmatrix}, \quad \hat{H}_0 = \begin{pmatrix} a(t) & c(t) \\ c(t) & b(t) \end{pmatrix} \tag{1.3}$$

and the superscript "T" denotes transpose.

The equations of motion for the vector $\underline{z}$ are obtained from the Poisson Brackets (P.B.) with the Hamiltonian (1.1). The relevant rules are easily derived. It is quite straightforward to realize that

$$\{\underline{z}, \underline{z}^T\} = \hat{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.4a}$$

and that

$$\{\underline{z}, \underline{z}^T \hat{B}\,\underline{z}\} = 2\hat{S}\hat{B}\,\underline{z} \;^* \tag{1.4b}$$

where $\hat{S}$ is the unit symplectic matrix in two dimensions. According to the above rules the equations of motion of $\underline{z}$ are easily written, namely,

$$\dot{\underline{z}} \equiv \{\underline{z}, \underline{z}^T \hat{H}\,\underline{z}\} = \hat{S}\hat{H}\,\underline{z} \tag{1.5}$$

and therefore immediately integrated, thus yielding

$$\underline{z}(t) = \hat{U}(t)\,\underline{z}(0) \tag{1.6}$$

where

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* Note that $\{\underline{z}^T, \underline{z}\} = 0$ and therefore the above product cannot be, strictly speaking, considered a conventional P.B. In the case that the second term in the $\{,\}$-product is a scalar function [as in Eq. (1.4b)] then the $\{,\}$ is a conventional P.B.

$$\hat{U}(t)=\left\{\exp\left[\hat{S}\int_0^t \hat{H}(t')dt'\right]\right\}_+ \tag{1.7}$$

and $\{,\}_+$ denotes time ordering for the classical evolution operator which is necessary when the commutator $[\hat{S}\hat{H}(t),\hat{S}\hat{H}(t')]$ is different from zero.

The above formalism is particularly useful to introduce the fluctuation tensor and its dynamical features, namely, let us pose

$$\hat{\Sigma}(t)=<\underline{z}(t)\underline{z}^T(t)>-<\underline{z}><\underline{z}^T> \tag{1.8}$$

where $<,>$ denotes ensemble average.

In a matrix form we recover the usual expression

$$\hat{\Sigma}(t)=\begin{pmatrix}\sigma_q^2 & \sigma_{pq}^2\\ \sigma_{pq}^2 & \sigma_p^2\end{pmatrix} \tag{1.9}$$

The equation of motion of $\hat{\Sigma}$ follows from Eq. (1.5) and reads

$$\frac{d}{dt}\hat{\Sigma}(t)=\hat{V}\hat{\Sigma}+\hat{\Sigma}\hat{V}^T \tag{1.10}$$

where $\hat{V}=\hat{S}\hat{H}$, and can be immediately integrated in terms of the above evolution operator $\hat{U}$ thus getting

$$\hat{\Sigma}(t)=\hat{U}(t)\hat{\Sigma}(0)\hat{U}^T(t) \tag{1.11}$$

Within this formalism it is easy to recover the well-known quadratic invariant of Courant-Snyder, namely, let us write the most general quadratic (time dependent) expression in $\underline{z}$

$$I=\underline{z}^T\hat{\Gamma}(t)\underline{z} \tag{1.12}$$

where

$$\hat{\Gamma}(t)=\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix} \tag{1.13}$$

Then we impose that I is a "time-dependent" invariant, i.e.,

$$\frac{dI}{dt}\equiv\frac{\partial I}{\partial t}+\{I,H\}=0 \tag{1.14}$$

thus obtaining the following equations specifying α,β,γ

$$\begin{cases} \dot{\beta} = -2b\alpha + 2c\beta & \beta(0) = \beta_0 \\ \dot{\gamma} = 2a\alpha - 2c\gamma & \gamma(0) = \gamma_0 \\ \dot{\alpha} = a\beta - \beta\gamma & \alpha(0) = \alpha_0 \end{cases} \tag{1.15}$$

from which it is easy to verify that

$$\beta\gamma = 1 + \alpha^2 \tag{1.16}$$

The Courant-Snyder invariant reads explicitly as

$$I = \gamma q^2 + 2\alpha p q + \beta p^2 \tag{1.17}$$

and it is widely used in the theory of linear transport.

2. TOWARDS A QUANTIZED THEORY

In the previous section we introduced the necessary background to derive a kind of uncertainty principle in the theory of linear transport. From Eq. (1.11) we get the relevant result

$$\det\hat{\Sigma}(t) = \det\hat{\Sigma}(0) \tag{2.1}$$

i.e., $\det\hat{\Sigma}$ is an invariant quantity and the emittance A of the system

$$A(t) = [<q^2><p^2> - <qp>^2]^{1/2} \tag{2.2}$$

is preserved in time.

Furthermore this means that

$$\sigma_q \sigma_p \geq A$$

which represents a kind of uncertainty principle in the canonical variables q, p and can be used as the starting point of our quantization procedure.

The rules are simple, the beam momentum is replaced by an operator specified by

$$p = \frac{\not{A}}{i}\frac{\partial}{\partial q} \tag{2.3}$$

where $\not{A}$ is the beam reduced emittance and the following rule of commutation is also assumed:

$$[p, q] = -i\not{A} \tag{2.4}$$

A "Hamiltonian" operator is finally associated to the longitudinal coordinate of propagation s

$$\hat{H} = i\lambda\!\!\!^{-}\frac{\partial}{\partial s} \tag{2.5}$$

so that the "Schrödinger" equation for a beam passing through a quadrupole of strength $k(s)$ reads

$$i\lambda\!\!\!^{-}\frac{\partial}{\partial s}\psi(q,s) = \left[-\frac{\lambda\!\!\!^{-2}}{2}\frac{\partial^2}{\partial q^2} + \frac{1}{2}k(s)q^2 \right]\psi(q,s) \tag{2.6}$$

where $\psi(q,s)$ denotes the beam wave function (b.w.f.). Clearly the b.w.f. $\psi(q,s)$ must be related in some way to the "classical" beam distribution $\rho(q,p,s)$ satisfying the Liouville equation for the time evolution of an ensemble of single-particle systems. The link is not obvious since ψ depends only on q and eventually on s while ρ is a function of q,p and s. The answer is given by the Wigner distribution function which is defined as follows:

$$W(q,p,s) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dy\,\psi^*\!\left(q + \frac{1}{2}y, s\right)\psi\!\left(q - \frac{1}{2}y, s\right)e^{ipy/\lambda\!\!\!^{-}} \tag{2.7}$$

and satisfies the Von-Neumann equation for a generic potential $V(q)$ (Ref. 6)

$$i\left[\frac{\partial}{\partial s} + p\frac{\partial}{\partial q}\right]W(q,p,s) = \lambda\!\!\!^{-\,-1}\left[V\!\left(q + \frac{i}{2}\lambda\!\!\!^{-}\frac{\partial}{\partial p}\right) - V\!\left(q - \frac{i}{2}\lambda\!\!\!^{-}\frac{\partial}{\partial p}\right)\right]W(q,p,s) \tag{2.8}$$

In the above-considered case of propagation through a quadrupole of strength $k(s)$ i.e., for an elastic potential $V(q) = \frac{1}{2}k(s)q^2$, the Von-Nuemann equation reduces to

$$\frac{\partial}{\partial s}W(q,p;s) = -\left[p\frac{\partial}{\partial q} - k(s)q\frac{\partial}{\partial p}\right]W(q,p;s) \tag{2.9}$$

which is equivalent to the Liouville classical equation. In this simple case we can identify the classical distribution density ρ with the Wigner distribution function W thus providing a completely consistent quantization scheme. Within this framework the physically meaningful quantity is not $|\psi(q,s)|^2 dq$ but $W(q,p;s)dqdp$ which ensures the consistency of our procedure.

As a simple exercise it is straightforward to see that

$$W(q,p;s)=\frac{1}{(2\pi)(\lambda\!\!\!^{-}/2)}\exp\left[-\frac{1}{2}\ \frac{\gamma(s)q^2+2\alpha(s)qp+\beta(s)p^2}{(\lambda\!\!\!^{-}/2)}\right] \tag{2.10}$$

is a solution of (2.9) if and only if the "time-dependent" parameters (α,β,γ) satisfy the following system of differential equations:

$$\begin{cases}\beta'=-2\alpha\\ \alpha'=k(s)\beta-\gamma\\ \gamma'=2k(s)\alpha\end{cases} \tag{2.11}$$

which are the well-known equations defining the evolution of the Twiss parameters in quadrupole lenses [see Eq. (1.15)].

3. CONCLUDING REMARKS

The extension of the developed theory to the case of nonlinear transport of charged beams is not straightforward and the problems involved can be illustrated in a simple example.

A sextupole term introduces, in the single-particle Hamiltonian, a contribution of the type

$$V(q,s)=\frac{\lambda(s)}{3}q^3 \tag{3.1}$$

Inserting the above potential in the Von-Neumann equation (2.8) we easily get the following evolution equation for the Wigner distribution function:

$$\frac{\partial}{\partial s}W(q,p;s)=-\underbrace{\left[p\frac{\partial}{\partial q}-\lambda(s)q^2\frac{\partial}{\partial p}\right]}_{\hat{L}}W(q,p;s)$$

$$-\frac{\lambda(s)}{12}\lambda\!\!\!^{-2}\frac{\partial^3}{\partial p^3}W(q,p;s) \tag{3.2}$$

where $\hat{L}$ denotes the Liouville operator associated to the potential (3.1) and given by $\hat{L}=-p\frac{\partial}{\partial q}+\lambda q^2\frac{\partial}{\partial p}$. The extra term in Eq. (3.2) is a purely "quantum" contribution and it is not present in the classical Liouville equation for $\rho(q,p;s)$. From this point of view W and ρ cannot be identified and since they coincide in the limit $\lambda\!\!\!^{-}\to 0$, ρ may be viewed as the "classical" counterpart of the Wigner distribution function W.

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